A CLASS OF PARTICULAR SOLUTIONS OF ONE-DIMENSIONAL NONLINEAR EQUATIONS OF HEAT CONDUCTION

V. N. Lagun

UDC 517.951:536.2

A class of solutions is obtained for the heat-conduction equations in the case of a power relation between the coefficient of thermal conductivity and the temperature.

Analytic solutions have not lost their topicality in spite of the fact that numerical methods for solving partial differential equations have attained a high level of development. Considerable attention is given to the solutions of nonlinear equations and, in particular, to the nonlinear equation of heat conduction. The results of solving a nonlinear equation of heat conduction can find application in diffusion or filtration problems depending on the physical sense of the sought function.

The profile of a nonlinear heat wave was obtained in [1]. In [2, 3] one-dimensional self-consistent solutions were found corresponding to the cases in which the amplitudes increasing as powers of time or heat flux were given. In [4] the self-consistent solution was obtained for the null value of the amplitude on the boundary. This has proved that for the solution to exist the first moment of the temperature must be constant. The results of [3, 4] for the two-dimensional case were extended in [5], where by introducing artificial parameters self-consistent solutions were obtained which correspond to all the possible variants for specifying the boundary conditions. A method for obtaining the general solution was proposed in [6]. There still remain the difficulties related to the finding of arbitrary functions appearing in general solutions from the given relations between the thermophysical parameters.

In the present article the solutions are found of one-dimensional nonlinear equations of heat conduction in the form of a sum of powers of the coordinate. The time-dependent coefficients are obtained in their general form.

The nonlinear one-dimensional heat-conduction equation

$$\frac{\partial T^s}{\partial t} = \frac{1}{r^\alpha}\frac{\partial}{\partial r}\left(r^\alpha a T^n \frac{\partial T}{\partial r}\right), \tag{1}$$

where α = 0, 1, 2 for the case of planar, axial, or central symmetry, respectively, is considered.

The nonlinearity of Eq. (1) prohibits the use of well-known methods of mathematical physics. In a number of cases nonstationary solutions are found in the class of generalized functions. Such solutions cannot be differentiable at two points of the heat flux: at the front of the heat wave $r_\alpha(t)$ and at the boundary r = 0.

It is known that in some sense the self-consistent solutions are limits of the non-self-consistent ones [7]. The analysis of self-consistent solutions [5] has enabled one to find out what transformations of Eq. (1) are necessary for the solutions to be sought in the form of polynomials.

The solution of (1) is sought in the form of a travelling wave on whose front $r_\alpha(t)$ the continuity conditions are satisfied for the temperature and for the heat flux:

$$T = aT^n \frac{\partial T}{\partial r} = 0. \tag{2}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 32, No. 3, pp. 508-511, March, 1977. Original article submitted February 10, 1976.

The function $r_\alpha(t)$ is found by solving

$$\frac{dr_\alpha}{dt} = -\lim_{r \to r_\alpha(t)} aT^{n-s} \frac{\partial T}{\partial r}. \tag{3}$$

Equation (3) was obtained for planar symmetry [1] but its form is maintained for other symmetries as well.

In (1) the sought function is modified. One sets

$$g(r, t) = T^{n-s+1}(r, t). \tag{4}$$

The partial differentiation will from now on be denoted by subscripts. For the function g(r, t) one obtains

$$\frac{s}{a} rg_t = r\left(gg_{rr} + \frac{s}{n-s+1} g_r^2\right) + \alpha gg_r. \tag{5}$$

The expression (3) now becomes

$$\frac{dr_\alpha(t)}{dt} = -\frac{a}{n-s+1} \lim_{r \to r_\alpha(t)} g_r. \tag{6}$$

The two equalities in (2) are reduced to a single one:

$$g\,|_{r=r_\alpha(t)} = 0. \tag{7}$$

One now replaces the space variable by

$$r = \eta^p. \tag{8}$$

Instead of (5) one obtains

$$\frac{s}{a} p^2 \eta^{2p-1} g_t = g\,[\eta g_{\eta\eta} - (\alpha p - p + 1)\, g_\eta] + \frac{s}{n-s+1} \eta g_\eta^2. \tag{9}$$

The solution of (9) is sought in the polynomial form:

$$g(\eta, t) = \sum_{i=0}^{m} A_i(t)\,\eta^i. \tag{10}$$

The degree m of the polynomial is related to the index p by

$$m = 2p, \tag{11}$$

that is, $m \geqslant 3$.

Having substituted (10) into (9) and carried out the required operations, one obtains from the comparison of the coefficients at the equal powers of η on both sides of the equality a system of 2m algebraic and differential equations for the (m + 1)-th coefficient of $A_i(t)$. In particular, by setting the coefficients of η^0 equal to zero one obtains

$$A_0 A_1 = 0, \tag{12}$$

which is required for solving two variants of the boundary conditions:

$$1.\ A_0 \neq 0,\ A_1 = 0, \tag{13}$$

$$2.\ A_0 = 0,\ A_1 \neq 0. \tag{14}$$

Having substituted (13) or (14) into the remaining 2m − 1 equations, one can see that the majority of the coefficients $A_i(t)$ vanish. There remains a system of differential equations whose number is at most three; it is used to determine the remaining unknown functions. The computational details are omitted and only the final results given; the following notation has been introduced to simplify them:

$$\tau = 2\alpha \frac{\alpha + 1 + 2v}{s} t + C_1; \tag{15}$$

$v = s/(n - s + 1)$; C_1 is an integration constant.

1. $A_1 = 0$, m is an odd integer. The solution is of the form

$$g(r, t) = A_0(t) + A_m(t)\, r^2, \tag{16}$$

where

$$A_0(t) = \frac{C_2}{\tau^{\frac{\alpha+1}{\alpha+1+2v}}}; \quad A_m(t) = -\frac{1}{\tau}. \tag{17}$$

The solution (16) with (17) satisfies the boundary condition: the heat flux is equal to zero. It has a physical sense for all forms of symmetry.

2. $A_1 = 0$, m is an even integer. The solution in the form of (10) is only possible for the case of planar symmetry; then it is

$$g(r, t) = A_0(t) + A_p(t)r + A_m(t)r^2, \tag{18}$$

where

$$A_0(t) = -\frac{C_3^2}{4}\frac{1}{\tau} + \frac{C_2}{\tau^{\frac{1}{1+2v}}}; \quad A_p(t) = \frac{C_3}{\tau}, \quad A_m(t) = -\frac{1}{\tau}. \tag{19}$$

The solution of (18) with (19) corresponds to the case in which on the boundary r = 0 neither the temperature nor the heat flux vanishes.

3. $A_0 = 0$. Formally, the solution is of the form

$$g(r, t) = A_1(t)r^{\frac{1}{p}} + A_m(t)r^2, \tag{20}$$

where

$$A_1(t) = \frac{C_2}{r^q}; \quad A_m(t) = -\frac{1}{\tau}; \quad q = \frac{2p^2(1+\alpha) + p(\alpha-1) + 1 + 4pv}{2p^2(1+\alpha+2v)}. \tag{21}$$

On the parameters s and n one imposes the constraint

$$v = p - 1 - \alpha p. \tag{22}$$

The constraint (22) has no physical sense in the case of axial symmetry, $\alpha = 1$.

In the expressions (17), (19), (21), etc., C_i are integration constants. By substituting the solutions (16), (18), (20) in (6) one finds the coordinates of the fronts of the heat waves as functions of time. One obtains, respectively,

$$r_\alpha(t) = C_3\tau^u, \quad \alpha = 0;\ 1;\ 2, \tag{23}$$

$$r_0(t) = \frac{C_3}{2} + C_4\tau^{\frac{s}{n+s+1}}, \tag{24}$$

$$r_\alpha(t) = \tau^{\frac{u}{2}}(C_2\tau^{-w} + C_3)^{\frac{p}{2p-1}}, \quad \alpha = 0;\ 2, \tag{25}$$

$$u = \frac{v}{\alpha+1+2v}, \quad w = \frac{v}{2p^2(1-\alpha)}.$$

The boundary condition (7) at the front is satisfied: for the solution (16), (17), (23) if $C_2 = C_3^2$; for (18), (19), (24) if $C_2 = C_4^2$; and for (20), (21), (25) if $C_3 = 0$. The arbitrary constants appearing in the solutions are determined from the known initial conditions.

Let us now consider the constraints which should be satisfied by the parameters n and s. For the solution (16) one has $n + 1 > s$ and also

$$T_r|_{r=r_\alpha(t)} = 0, \quad n < s < n+1,$$

$$0 < T_r[r_\alpha(t), t] < \infty, \quad n = s,$$

$$T_r \underset{r \to r_\alpha(t)}{\to} \infty, \quad n > s.$$

One has, in addition, in the case of central symmetry, $3(n + 1) < s$.

At the front one has

$$T_r \underset{r \to r_\alpha(t)}{\to} \infty.$$

The solution (19) with planar symmetry makes sense for $n > s$. The temperature gradient at the front is infinite.

Finally, in the solution (20) the physical parameters n and s must satisfy the equalities

$$\frac{n+1}{s}=\frac{m}{m-2},\ \alpha=0,\quad \frac{n+1}{s}=\frac{m}{m+2},\ \alpha=2.$$

NOTATION

T, temperature; r, coordinate; t, time; a, dimensional parameter; s, n, dimensionless parameters.

LITERATURE CITED

1. Ya. B. Zel'dovich and A. S. Kompaneets, "Theory of heat propagation for temperature-dependent heat conduction," in: Collected Papers in Honor of A. F. Ioffe's 70th Birthday [in Russian] (1950).
2. G. I. Barenblatt, Prikl. Mat. Mekh., 16, No. 6 (1952).
3. G. I. Barenblatt, Prikl. Mat. Mekh., 17, No. 6 (1953).
4. G. I. Barenblatt and Ya. B. Zel'dovich, Prikl. Mat. Mekh., 21, No. 5 (1957).
5. V. N. Lagun, Inzh.-Fiz. Zh., 28, No. 4 (1975).
6. B. M. Berkovskii, Dokl. Akad. Nauk BelorusSSR, 13, No. 3 (1969).
7. Ya. B. Zel'dovich and Yu. P. Raizer, Physics of Shock Waves and High Temperature Hydrodynamic Phenomena, Academic Press (1966, 1967).
8. A. A. Samarskii and I. M. Sobol', Zh. Vychisl. Mat. Mat. Fiz., 3, No. 4 (1963).

EFFECT OF PERIODIC SYSTEM OF NARROW INCLUSIONS ON A PLANE STEADY TEMPERATURE FIELD

I. M. Abdurakhmanov and B. G. Alibekov

UDC 536.24.02

Finding the complex potential of a plane temperature field perturbed by a periodic system of narrow inclusions reduces to solving a singular integrodifferential equation. The effect of cracks on an arbitrary periodic temperature field is considered.

1. Let a plane periodic (period $2a$) steady temperature field determined by the harmonic function $T_0(x, y) = \operatorname{Re} F(t)$ be perturbed by a $2a$-periodic system of narrow macroinclusions of a different material or cracks. For approximate formulation of the problem and its effective solution, we take the narrow inclusions as lines in the complex z plane. To be specific, we assume that the thermal conductivity of the inclusions k_0 is considerably less than that of the main medium (the body) k, i.e., $k_0 \ll k$.

Isolating in the z plane a band of width $2a$ $(-a \leqslant x \leqslant a)$, we denote the narrow inclusions present in the band, taken in any order, by Γ_n, n = 1, 2, ..., N. We denote the set of all the lines Γ_n by Γ, i.e., $\Gamma = \Gamma_1 + \ldots + \Gamma_N$.

The problem is to find the complex potential of the periodic temperature field perturbed by the inclusions, $W(z) = T + i\psi$; T is the temperature and ψ the current function.

We write W(z) as the sum of the potential of the temperature field of the homogeneous medium (without inclusions) F(z) and integrals of Cauchy type taken along the line Γ and all of the congruent lines, i.e., we write

$$W(z)=F(z)+\Phi(z),\quad \Phi(z)=\frac{1}{4ai}\int_{\Gamma}\omega(t)\operatorname{ctg}\frac{\pi(t-z)}{2a}dt. \tag{1.1}$$

Dagestan Branch, Academy of Sciences of the USSR. Institute of Physics, Makhachkala.

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 32, No. 3, pp. 512-515, March, 1977. Original article submitted March 30, 1976.